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A Corrector for Probability Dilution in Satellite Conjunction Analysis

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This paper introduces the problem of probability dilution in satellite conjunction analysis to the wider uncertainty quantification community and proposes random-to-fuzzy transformation as a practical fix. Probability dilution is presented as an "ostrich paradox," a situation in which poor data quality appears to make a system safer, an absurd conclusion. Plausibility of collision derived from a judiciously chosen random-to-fuzzy transformation provides a more intuitive metric for the possibility of a collision between two satellites. Two candidate transformations are presented, provisionally labeled "red fuzzy" and "blue fuzzy." One of these transformations (blue) emulates the current maximum probability approach. The other transformation (red) yields clear indications as to when the user is in the probability dilution region. The paper concludes with a case in favor of the red random-to-fuzzy transformation as the blueprint for a usable full-scale solution.

I. Introduction

A. An Ostrich Paradox

Computing the probability of collision between two satellites yields an interesting phenomenon: probability dilution. Given a fixed mean closest approach distance between two satellites, past a certain point, as the variance in the position estimate increases, the probability of collision decreases.[1] That decrease, illustrated in Figures 1-2, is probability dilution.

Since the probabilities under investigation are epistemic rather than aleatory, variance is driven by data quality, not randomness in the satellite paths. This leads to the absurd implication that as data quality decreases, the safety of the satellites being modeled increases. This is a real-world Ostrich Paradox - a situation in which lack of knowledge creates the illusion of safety.

Epistemology drives this paradox, not mathematics. In trying to prevent a collision between two satellites, one is metaphorically trying to avoid threading a needle. Consider the absurd situation in which one is literally trying to avoid threading a needle. Given a fixed mean miss distance, the shakier the hand, the more likely one is to successfully *not* thread the needle. Probability dilution makes sense in the needle threading problem. In contrast, the idea that lower quality models or lower quality data make a satellite safer is foolish on its face. The difference is that in the needle-threading case, hand tremors represent an aleatory uncertainty; in satellite collision, the uncertainty is epistemic in nature.

A plethora of competing approaches for dealing with epistemic uncertainty currently exist. These approaches include but are not limited to subjective Bayesian probability,[2,3] classical fuzzy theory,[4] intervals,[5,6] confidence distributions,[7–10] fiducial distributions,[11,12] Dempster-Shafer theory (subjective-flavor[13,14] or fiducial-flavor[15,16]), and a Dempster-Shafer inspired random-fuzzy hybrid.[17] No approach has yet emerged as the dominant norm. So, in trying to address or avoid the problem of probability dilution, the practitioner has a varied toolkit but only provisional theoretical guidance.

That being said, the problem still needs to be resolved. And there is no reason to think a practical fix cannot be developed. Practice often outpaces theory.

B. Computing Collision Probability

An overview of methods for computing the probability of collision between two satellites is provided by Alfano.[18] Alfano's approach involves four simplifying assumptions:

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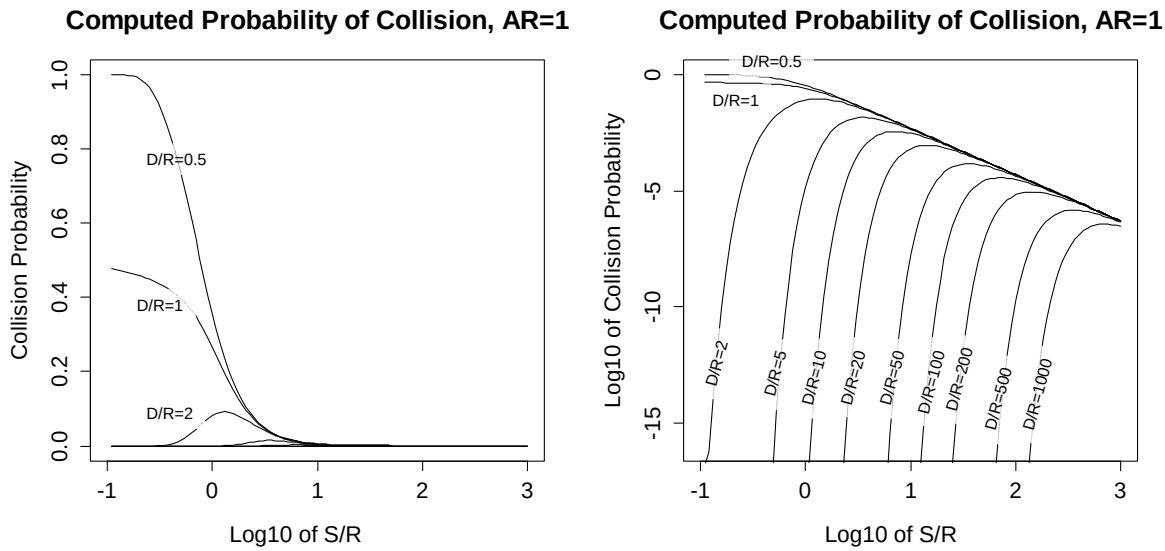

Figure 1. Computed Collision Probability as a Function of D/R and S/R, where D is nominal distance between satellites at closest approach, R is combined radius of the two satellites, S is the standard deviation of the relative position of the two satellites at closest approach, and AR=1 refers to the fact that standard deviation is equal in both directions, i.e. $\sigma'_x = \sigma'_y$.

1. Only uncertainty in the position of the satellite at expected closest approach is considered.
2. Position uncertainty is assumed to have a multivariate Gaussian distribution.
3. The relative velocity of the two satellites is high enough at closest approach that uncertainty in the direction of the relative velocity vector can be neglected. The problem therefore reduces to two dimensions.
4. The satellites are modeled as spheres.

The calculation of collision thereby simplifies to the calculating the probability of (x, y) such that

$$x^2 + y^2 \leq R^2$$

where R is the sum of the radii of the two satellites and (x, y) is the uncertain displacement of one satellite with respect to each other at closest approach in the plane perpendicular to the relative velocity vector at closest approach. Let (x_0, y_0) be the expected coordinates at closest approach. The probability distribution of (x, y) has an elliptical Gaussian distribution given by the following density relation:

$$f(x, y) \propto \exp\left(-\frac{1}{2}\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}^T \Sigma_{XY}^{-1} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}\right)$$

where Σ_{XY} is the covariance matrix for x and y.

These coordinates can be transformed to align with the covariance matrix, which is achieved via the following formula:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = E_\Sigma^T \begin{bmatrix} x \\ y \end{bmatrix}$$

where E_Σ is the matix normalized column eigenvectors of the covariance matrix. The resulting probability distribution for (x', y') is as follows:

$$f(x, y) = \frac{1}{\sigma'_x \sigma'_y 2\pi} \exp\left[-\frac{1}{2}\left(\left(\frac{x' - x'_0}{\sigma'_x}\right)^2 + \left(\frac{y' - y'_0}{\sigma'_y}\right)^2\right)\right]$$

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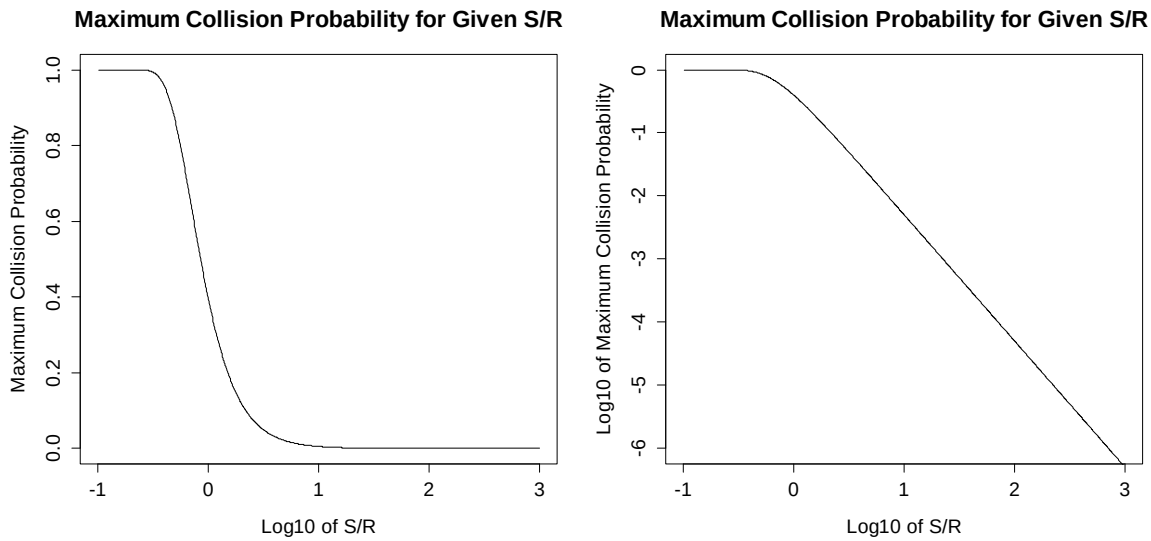

Figure 2. Maximum Computable Probability of Collision, given S/R. This occurs at D/R=0.

where $\sigma_x'^2$ and $\sigma_y'^2$ are the eigenvalues of the covariance matrix, Σ_{XY}, and the probability of collision is the probability of the event:

$$x'^2 + y'^2 \leq R^2.$$

The Alfano approach to computing collision probability usually shifts the origin to the center of the probability ellipse, with no additional transformations.

The work presented in this paper takes a slightly different tack, rewriting x' and y' as follows:

$$x' = x_0' + \sigma_x' \xi_1$$
$$y' = y_0' + \sigma_y' \xi_2$$

where ξ_1 and ξ_2 are independent unit normal variates. The probability of collision can be computed by taking the integral of the following ellipse:

$$\left(\frac{\xi_1 + x_0'/\sigma_x'}{R/\sigma_x'}\right)^2 + \left(\frac{\xi_2 + y_0'/\sigma_y'}{R/\sigma_y'}\right)^2 \leq 1$$

in a circular unit 2D normal space centered on the origin, which has the following weight function:

$$f\left(\xi_1, \xi_2\right) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)\right].$$

The difference between these two probability calculation approaches is illustrated in Figure 3. One advantage of the rearrangement utilized in this paper is that it clarifies that collision probability is a function of the ratios between mean offset distance, satellite size, and standard deviation of offset between the two satellites.

After this rearrangement, probability is computed according to the method stipulated by Patera:[19]

$$\text{Collision Probability} = \int_0^{2\pi} \left(\frac{1 - e^{-r^2/2}}{2\pi r^2}\right) \left(\frac{R^2}{\sigma_x'\sigma_y'} - \frac{x_0' R}{\sigma_x'\sigma_y'}\cos\psi - \frac{y_0' R}{\sigma_x'\sigma_y'}\sin\psi\right) d\psi$$

where

$$r^2 = \left(-\frac{x_0'}{\sigma_x'} + \frac{R}{\sigma_x'}\cos\psi\right)^2 + \left(-\frac{y_0'}{\sigma_y'} + \frac{R}{\sigma_y'}\sin\psi\right)^2.$$

This integral can be approximated using the trapezoidal rule to negligible error using

$$n_{pts} \geq 10 \max\left(\frac{\sigma_x'}{\sigma_y'}, \frac{\sigma_y'}{\sigma_x'}\right)$$

evenly space points.

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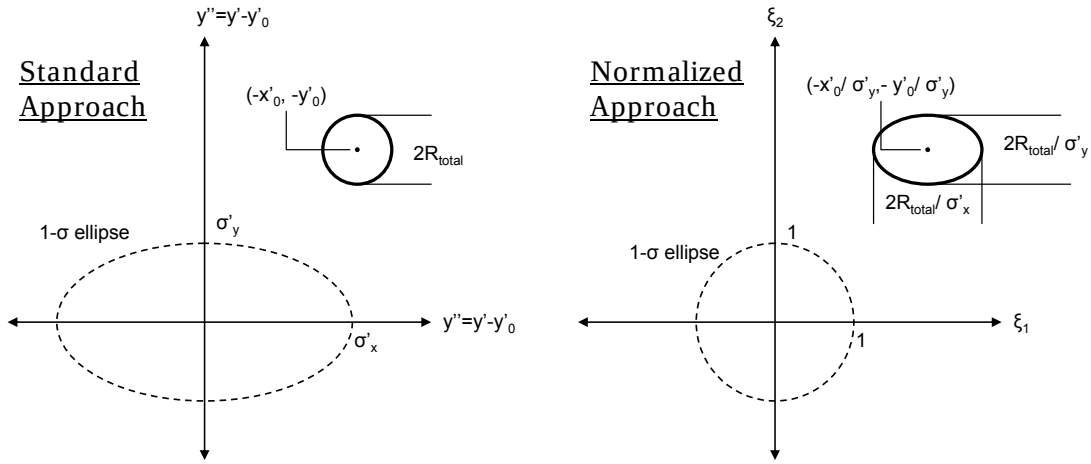

Figure 3. Different Methods for Describing Collision Probability

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C. Fuzzy Variables and Random-to-Fuzzy Transformation

The fix to probability dilution proposed in this paper is based on the development and application of an appropriate random-to-fuzzy transformation. Fuzzy variables are less well-known than random variables, and random-to-fuzzy transformations are even less well-known. Therefore, a quick review of these concepts is in order.

Fuzzy variables are to possibility theory as random variables are to probability theory. In possibility theory, two set functions traditionally called necessity and possibility represent uncertainty in an event. For this paper, the more intuitive (and logically equivalent) terms "plausibility" and "confidence" are used. In possiblity theory, plausibility and confidence are governed by the following axiomatic relations:[17]

$$\mathrm{Conf}\left(A\right) \geq 0 \quad \forall A \subset \Omega$$

$$\mathrm{Conf}\left(\Omega\right) = 1$$

$$\mathrm{Conf}\left(A \cap B\right) = \min\left[\mathrm{Conf}\left(A\right), \mathrm{Conf}\left(B\right)\right]$$

$$\mathrm{Plaus}\left(A\right) = 1 - \mathrm{Conf}\left(A^C\right)$$

where Ω is the set of all possible values and A^C is the complement of A. All of the possibilistic assessments presented in this paper derive from a conservative random-to-fuzzy transformation, which means plausibility and confidence levels can be treated as upper and lower bounds on probability (respectively).[a] Figure 4 provides a quick guide for interpreting possibilitistic representations of uncertainty.

Practically speaking, fuzzy variables are represented using pointwise possibility functions, which are analogous to the probability mass or probability density functions on random variables.[4,17] The possibility function, $r : \Re \rightarrow [0, 1]$, for a fuzzy variable outputs values from zero to one, has a guaranteed maximum value of one, and is strictly non-decreasing prior to achieving that maximum and strictly non-increasing after that maximum. Plausibility and confidence values are computed from the possibility function as follows:

$$\mathrm{Plaus}_X\left(A\right) = \max\left\{r\left(x\right) : x \in A\right\}$$

and

$$\mathrm{Conf}_X\left(A\right) = 1 - \max\left\{r\left(x\right) : x \in A^C\right\}$$

where A^C is the complement of A.

Salicone introduced the idea of a random-to-fuzzy transformation, in which a probability density function or cumulative distribution function is converted into a possibility function. In her book on Evidence Theory,[17] Salicone suggests a median-anchored random to fuzzy transformation, which can be described as follows:

$$r_X\left(x\right) = 2\min\left[F_X\left(x\right), 1 - F_X\left(x\right)\right]$$

[a]Orthodox Possibility Theory adherents would usually balk at such an interpretation.

Plausibility of / Confidence in Collision	Interpretation
Plaus = 1 Conf large	Satellite definitely in danger
Plaus = 1 Conf small	Need better data
Plaus large Conf = 0	Need better data
Plaus small Conf = 0	Satellite safe

Figure 4. Interpretation Guide for Possibilistic Representations of Collision Uncertainty

where F_X is the cumulative distribution function for x. Salicone's original intent was to apply this transform as part of the construction of a compound random-fuzzy variable. That general scheme is abandoned for the purposes of this paper, and the idea of a random-to-fuzzy transformation is adopted in isolation as an ad-hoc fix to the problem of probability dilution. Similarly, this paper's use of a random-to-fuzzy transformation should not be taken as a general endorsement of fuzzy theory or the rules of fuzzy propagation as stipulated by Zadeh[4] or as suggested by Salicone.[17]

D. Current Approach

The current approach to probability dilution is to insist on data that are precise enough to not incur probability dilution. Failing that, the maximum probability (maximization taking place over position uncertainty, given a fixed distance and satellite radius) is used.[1] These approaches are potentially infeasible and definitely questionable, respectively.

Better data are not always available. Precision in the measurement of satellite position and velocity is constrained by the hardware available. A satellite operator has to design and execute a collision avoidance maneuver within hours or days. Satellite tracking hardware is improved over years and decades. "Get better data" is only an option if there is better hardware in place than the hardware initially used to alert the operator to a potential problem. And even then, the best tracking data available may not be sufficiently precise to eliminate probability dilution. Moreover, the models used to predict orbit perturbations also suffer epistemic uncertainties that are not easily reduced.

The other current approach, to simply assume the maximum collision probability for the nominal miss distance and combined satellite size, has no more theoretical justification than using (for example) the maximum collision probability for the given uncertainty and combined satellite size (see Figure 2). While the first option is more intuitively appealing, it has no better theoretical foundation than the second. Neither addresses the fundamental problem of probability dilution - the clash between the implications of applying probability theory to epistemic uncertainty in satellite collision analysis ... and basic common sense.

E. Other Approaches

At least two efforts have been made to correct the problem of probability dilution in satellite conjunction analysis. Both of these attempts were made from the same position taken in this paper: that probability dilution renders probability of collision an inappropriate metric for collision risk. The first of these two attempts was made by Clayton Frigm, who attempted to derive a tunable collision risk metric, the F-value,

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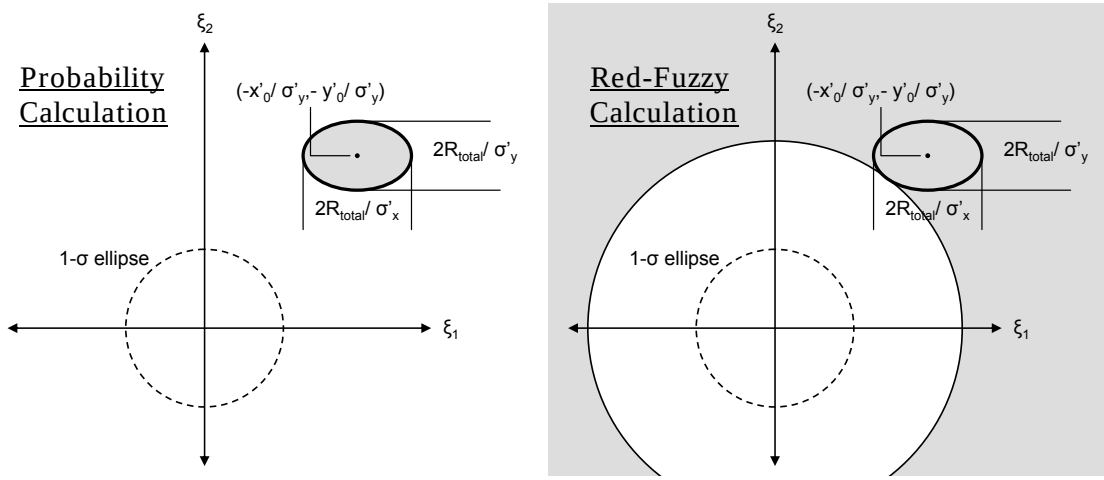

Figure 5. Calculation of Probability vs Calculation of Red-Fuzzy Plausibility, Both Integration Regions Marked in Gray

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that weighted contributions from risk metrics such as collision probability, distance at closest approach, and satellite position uncertainty.[20] Unfortunately, results derived from an experimental implementation of this metric were not appreciably different from those derived from collision probability on its own.[21] The other effort - by Sun, Luo, and Niu - involved the creation of a piecewise metric for collision risk, which transitions from a distance measure to the ratio of collision probability to a probability-like warning threshold that vanishes as position uncertainty increases.[22] The result is a metric that successfully eliminates dilutive behavior, albeit with a somewhat ambiguous quantitative interpretation. As discussed in Section VI, the approach favored in this paper suffers its own ambiguities but also provides the blueprint for a less ambiguous fix.

II. Proposed Methodology

Two distinct random-to-fuzzy transformations are proposed here as alternative approaches for resolving the probability dilution problem. Each involves transforming either the displacement or distance at closest approach from a probabilistic to a possibilistic representation of uncertainty. For the purposes of this paper, these are labeled the "red fuzzy" and "blue fuzzy" transformations. The "blue fuzzy" transformation is based on a modification of Salicone's median-anchored transformation, applied to the uncertain distance between the two satellites.[b] The "red fuzzy" transformation is a likelihood-based approach applied in the shifted, normalized displacement space, (ξ_1, ξ_2). Before these transformations are exposited, however, the rationale for a random-to-fuzzy transformation as a fix for probability dilution is provided.

A. Understanding the Ostrich Paradox

Like all paradoxes, the Ostrich Paradox boils down to a tension between a strong intuition and a logically sound deduction from premises that initially seem reasonable. If one is to side with the intuition, then one of the premises supporting the conflicting deduction must be rejected. In this paper, the Ostrich Paradox is broken down as follows:

- Deduction - As variance increases, the probability of not threading a needle also increases.
- Conflicting Intuition - Risk never decreases as the result of an increase in epistemic uncertainty.
- Faulty Premise - Probability theory is the appropriate framework for epistemic uncertainty.

That probability theory is not the correct framework does not necessarily imply that possibility theory is. However, possibility theory does have an attractive quality. For a fuzzy variable with a fixed central value, as

[b] A generalization encompassing both transformations can be found in Tore Schweder's 2007 paper on confidence nets, which are essentially upside-down fuzzy variables giving confidence intervals.[23]

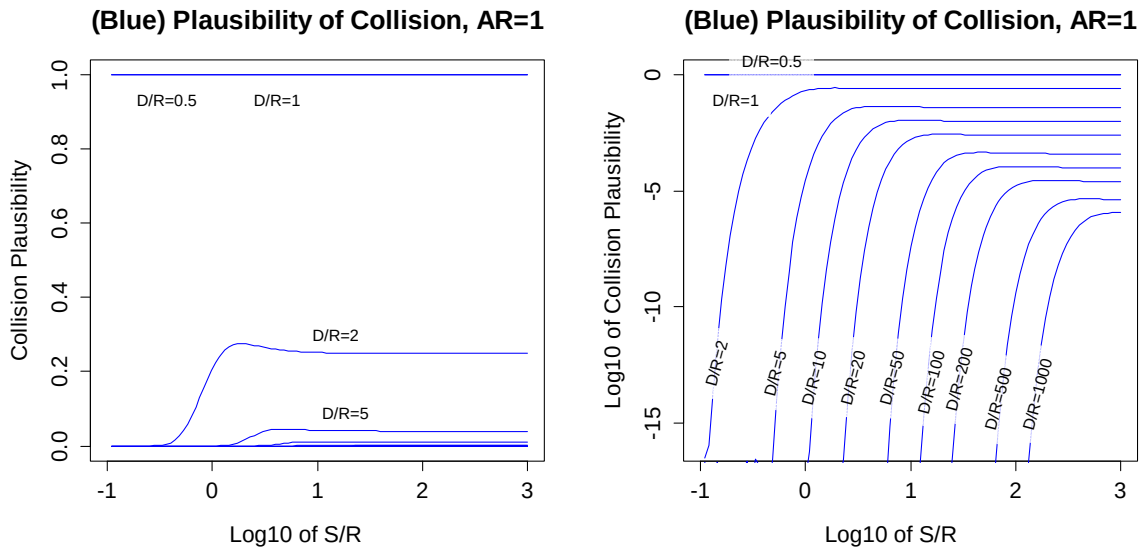

Figure 6. Collision Plausibility (Blue Variant) as a Function of D/R and S/R

uncertainty increases, plausibility in any proposition increases (or at least does not decrease). If plausibility of failure is taken as a risk metric, it satisfies the strong intuition that drives the Ostrich Paradox. So, while there is no a priori guarantee that any conservative random-to-fuzzy transformation will resolve an Ostrich Paradox, the strategy is at least worth a shot.

B. Blue Fuzzy Transformation

The blue fuzzy transformation starts with propagation of uncertainty from (x', y') plane to distance, d, defined as follows:

$$d = \sqrt{x'^2 + y'^2}$$

where x' and y' are defined as in Section I.B. Taking the CDF of this quantity, the blue fuzzy transform is then defined as follows:

$$r_\Delta(d) = \begin{cases} F_\Delta(d) / F_\Delta(D), & \text{if } d \leq D \\ (1 - F_\Delta(d)) / (1 - F_\Delta(D)), & \text{if } d \geq D \end{cases}$$

where $D = \sqrt{x_0'^2 + y_0'^2}$ is the map of the expected closest approach coordinates and F_Δ is the cumulative distribution function for d. The plausibility of collision is computed as follows:

$$\text{Blue Plausibility of Collision} = \max\{r_\Delta(d) : d \leq R\} = \begin{cases} \frac{F_\Delta(R)}{F_\Delta(D)}, & \text{if } R \leq D \\ 1, & \text{if } R \geq D \end{cases}$$

Results obtained using the blue-fuzzy transformation are given in Figure 6.

C. Red Fuzzy Transformation

The red fuzzy transformation is computed in the (ξ_1, ξ_2) plane. In this transform, the possibility value attached to any (ξ_1, ξ_2) point is the probability of randomly drawing a less likely (ξ_1, ξ_2) pair than the pair under inspection. Mathematically, this translates the following formula:

$$r_{\Xi_1,\Xi_2}(\xi_1, \xi_2) = \exp\left[-\frac{1}{2}\left(\xi_1^2 + \xi_2^2\right)\right].$$

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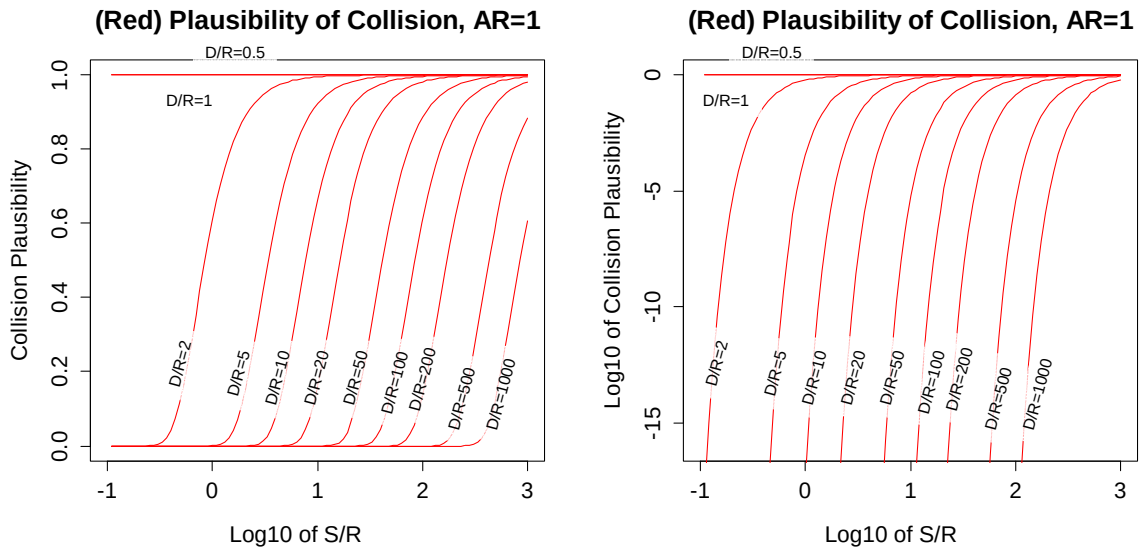

Figure 7. Collision Plausibility (Red Variant) as a Function of D/R and S/R

The resulting plausibility of collision is computed as follows:

$$\text{Red Plausibility of Collision} = \max\left\{ r_{\Xi_1,\Xi_2}(\xi_1,\xi_2) : \left(\frac{\xi_1 + x_0'/\sigma_x'}{R/\sigma_x'}\right)^2 + \left(\frac{\xi_2 + y_0'/\sigma_y'}{R/\sigma_y'}\right)^2 \leq 1 \right\} = \exp\left[-\frac{1}{2}\hat{\rho}^2\right]$$

where $\hat{\rho}$ is the smallest distance from the origin to the ellipse in the normalized calculation of collision probability. Note that if the ellipse covers the origin, equivalently D/R ≤ 1, then $\hat{\rho} = 0$ and the plausibility of collision resulting from the red fuzzy transformation is equal to one.

III. Demonstration Approach

The performance of the red fuzzy and blue fuzzy transformations is illustrated by examination of curves giving probability/plausibility as a function of S/R, the ratio of uncertainty to combined satellite size, for fixed values of D/R, the ratio of expected offset to combined satellite size. These plausibilities are derived using the same assumptions as in Section I.B. That is, a fast approach leading to a linear velocity treatment and resulting reduction of the problem to two dimensions. For the purpose of these charts, $AR = 1$; that is, combined standard deviation in the x' and y' directions is equal. This assumption simplifies the demonstration, but in practice, it will almost certainly not be satisfied.

As a point of reference, the demonstration starts with the computed probability curves in Figure 1. Next, results from the two proposed random-to-fuzzy transformations are illustrated in Figures 6-7. All three approaches are then plotted together for D/R = 2 and D/R = 100 in Figures 8-9, respectively.

IV. Results

A. General Behavior of Red Fuzzy and Blue Fuzzy Transformations

The red fuzzy and blue fuzzy transformations provide distinct approaches to the probability dilution problem. Blue plausibility tracks proportionally with the maximum collision probability approach, with some deviation after maximum collision probability is achieved as a function of relative uncertainty (S/R). Past this point, blue plausibility converges to an asymptotic value two to ten times as large as the maximum probability.

The results from the red fuzzy transformation are more promising. Prior to the maximum collision probability point, in the dilution-free region, red plausibility tracks with blue plausibility and collision probability; although, it is always greater than either. As relative uncertainty approaches the maximum collision probability point, red plausibility begins to diverge from the other two representations of collision risk. Finally, as

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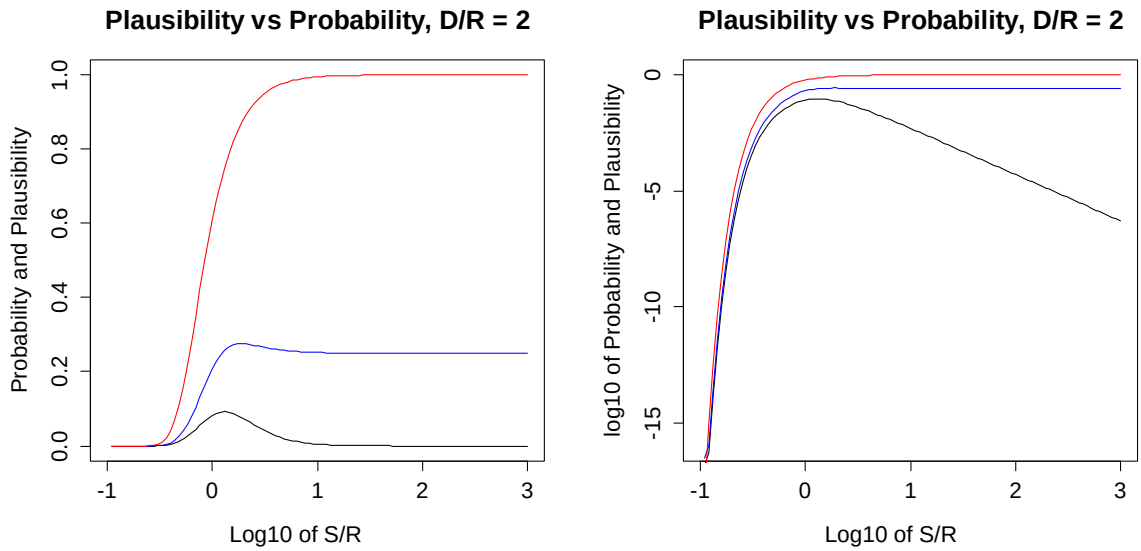

Figure 8. Comparative Behavior of Probability and Plausibilities for D/R = 2

relative uncertainty continues to grow, plausibility of collision from the red fuzzy transformation approaches unity. This is exactly the qualitative behavior one would expect from a reasonable representation of collision risk in the face of probability dilution.

B. Asymptotic Proofs

The asymptotic behavior of the red fuzzy and blue fuzzy transformations observed in Figures 6-7 can be confirmed for all aspect ratios and offsets. First, note that if $D/R \leq 1$, red plausibility and blue plausibility of collision are always equal to one; so, only the $D/R > 1$ case need be considered in detail.

Given $D/R > 1$, blue plausibility of collision is equal to the ratio of the probabilities of (ξ_1, ξ_2) falling within two ellipses, both centered at $\left(-\frac{x_0'}{\sigma_x'}, -\frac{y_0'}{\sigma_y'}\right)$ the larger with semi-major and semi-minor axes as $\frac{D}{\sigma_x'}$ and $\frac{D}{\sigma_y'}$, the smaller with semi-major and semi-minor axes as $\frac{R}{\sigma_x'}$ and $\frac{R}{\sigma_y'}$. As standard deviation increases, given a fixed $\frac{\sigma_x'}{\sigma_y'}$ and (x_0', y_0'), both ellipses shrink, eventually to the point that the probabilities can be approximated as follows:

$$\lim_{S \to +\infty} \left[F_\Delta\left(R\right)\right] = f_{\Xi_1,\Xi_2}\left(\frac{-x_0'}{\sigma_x'}, \frac{-y_0'}{\sigma_y'}\right) \pi \left(\frac{R}{\sigma_x'}\right)\left(\frac{R}{\sigma_y'}\right)$$

and

$$\lim_{S \to +\infty} \left[F_\Delta\left(D\right)\right] = f_{\Xi_1,\Xi_2}\left(\frac{-x_0'}{\sigma_x'}, \frac{-y_0'}{\sigma_y'}\right) \pi \left(\frac{D}{\sigma_x'}\right)\left(\frac{D}{\sigma_y'}\right).$$

The asymptotic value for blue plausibility is as follows:

$$\lim_{S \to +\infty} \left[r_\Delta\left(R\right)\right] = \lim_{S \to +\infty} \left[\frac{F_\Delta\left(R\right)}{F_\Delta\left(D\right)}\right] = \left(\frac{R}{D}\right)^2$$

As illustrated, unless $D/R \leq 1$, the blue plausibility of collision will converge to a value less than one. Whether the relationship of this asymptotic value to maximum probability is maintained is a question left to future work.

The proof that the red plausibility of collision always converges to one is slightly more complicated. First note that

$$0 \leq \hat{\rho} \leq \sqrt{\hat{\xi}_1^2 + \hat{\xi}_2^2}$$

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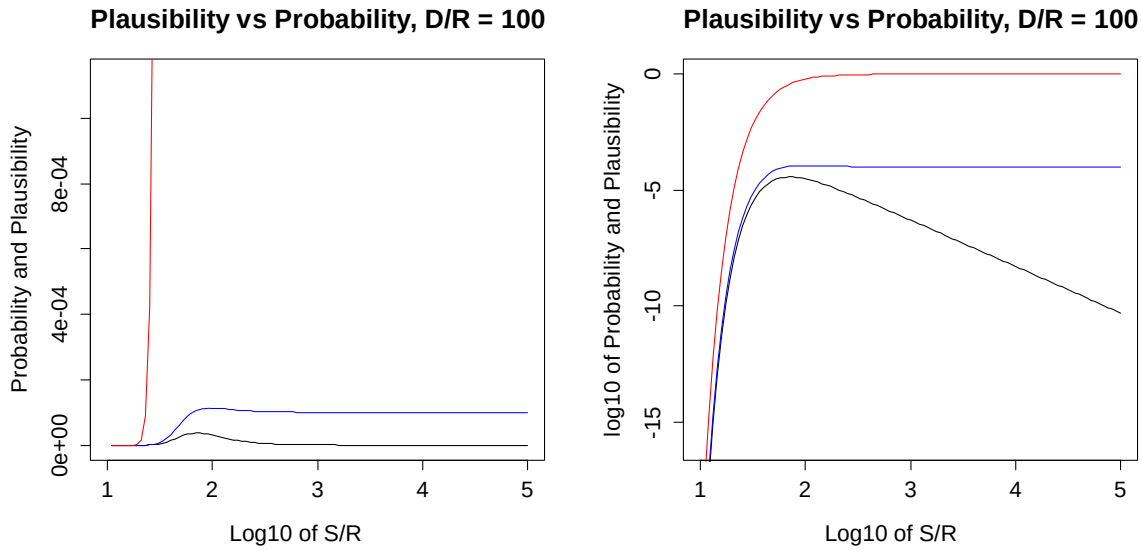

Figure 9. Comparative Behavior of Probability and Plausibilities for D/R = 2

where $\left(\hat{\xi}_1, \hat{\xi}_2\right)$ solves the following two constraints:

$$\hat{\xi}_2 = \left(\frac{y_0' \sigma_x'}{x_0' \sigma_y'}\right) \hat{\xi}_1 \quad \text{and} \quad \left(\sigma_x' \hat{\xi}_1 + x_0'\right)^2 + \left(\sigma_y' \hat{\xi}_2 + y_0'\right)^2 = R^2.$$

With a little work, it can be shown that

$$0 \leq \hat{\rho} \leq \begin{cases} \sqrt{1 + \left[\left(\frac{\sigma_x'}{\sigma_y'}\right)^2 - 1\right]\left(\frac{y_0'}{D}\right)^2} \left(\frac{D-R}{\sigma_x'}\right), & \text{if } \sigma_x' \geq \sigma_y' \\ \sqrt{1 + \left[\left(\frac{\sigma_y'}{\sigma_x'}\right)^2 - 1\right]\left(\frac{x_0'}{D}\right)^2} \left(\frac{D-R}{\sigma_y'}\right), & \text{if } \sigma_x' \leq \sigma_y'. \end{cases}$$

Phrased more generally,

$$0 \leq \hat{\rho} \leq K\left(\frac{D-R}{S}\right)$$

where $K \geq 1$ is a function of probability density aspect ratio, σ_x'/σ_y', and the nominal offset, (x_0', y_0'). As S approaches infinity, $\frac{D-R}{S}$ drops to zero. So,

$$\lim_{S \to +\infty} [\hat{\rho}] = 0,$$

which means that as S approaches infinity, the red plausibility of collision approaches one, for all nominal offsets and all probability aspect ratios. That guaranteed convergence to one as uncertainty grows remains one of the most attractive features of red plausibility as an indicator of collision risk.

V. Conclusions

A. Overview

This paper introduced a potentially workable fix to the problem of probability dilution in satellite conjunction analysis. The concept of an Ostrich Paradox, a situation in which lack of knowledge appears to make a system safer, was introduced. Next, the calculation of satellite collision probability was reviewed, as were the basics of possibility theory and random-to-fuzzy transformations. The exact nature of the probability dilution problem was discussed. The "blue fuzzy" and "red fuzzy" transformations were then proposed as

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potential fixes for probability dilution. It was found that the red fuzzy transformation demonstrates behavior more desirable as a representation of collision risk than the behavior demonstrated by computed collision probability.

B. Preference for Red Plausibility

The preference for "red plausibility" as a metric for collision risk assessment is based on a simple principle: a system dominated by epistemic uncertainty is safe only when the behavior of each component is known and known to a high degree of confidence to not cause problems. In the case of satellite collision avoidance, that means a low probability of collision *and* little overlap between the two uncertain satellite paths. If collision probability is low merely due to high variance, then overlap is large and the system is not safe. Plausibility as given by the "red fuzzy" transformation in Section II.C is an attractive metric for collision risk because it emulates collision probability when overlap is small, and it converges to one (i.e. high risk) when overlap is large.

C. Implications Beyond the Ostrich Paradox

Ostrich Paradoxes and the fix for them demonstrated in this paper have implications beyond satellite collision analysis. For years, engineers have been applying probability theory to epistemic uncertainty, with varying degrees of success. The author of this paper has himself advocated for probabilistic representations of epistemic uncertainty, when justifiable in terms of coverage probability (i.e. statistical confidence).[9,10] However, an Ostrich Paradox cannot be explained in terms of the usual subjective Bayesian culprits. Even if the uncertainties in satellite position were completely objective and phrased in terms of prior-free confidence distributions, probability dilution would still be an issue. Probability dilution represents an interaction solely between the data quality and the epistemic probability of disaster, an interaction *so* counter-intuitive that it gives the practitioner a legitimate reason to question and even reject the mathematically correct probabilistic result. This will be true for any Ostrich Paradox. Similarly, it may be hoped that a judiciously chosen random-to-fuzzy transformation may provide a fix for other Ostrich Paradoxes in the future.

The problem of probability dilution in satellite conjunction analysis, together with the fix presented in this paper, constitute circumstantial evidence for the argument that epistemic uncertainty should be represented using fuzzy variables. Possibility theory has been asserted as the normative framework for representing epistemic uncertainty,[17] but the author of this paper is unaware of a work in which a comprehensible and principled argument for that standard has been made. There is at present no epistemologically compelling theory which would argue for the necessity of the fix suggested in this paper. However, the analysis provided in this paper may be an early step towards building such a theory.

VI. Future Work

The red fuzzy transformation preferred in this paper may not be a usable metric for satellite collision risk, but rather may serve as a blueprint for getting to one. Red plausibility has the right qualitative behavior. However, quantitatively, one can only say for it that it is conservative with respect to probability of collision. Conservatism with respect to a rejected metric is not a conclusive argument in favor of the quantitative correctness of a proposed metric.[c] The red fuzzy transformation converts uncertainty in a reduced two-dimensional relative displacement space, in which any originating confidence distributions are likely to have been scrambled.[d] If the conversion from random to fuzzy were made in the original twelve-dimensional problem space (i.e. the two six-element satellite states), it should be that the pointwise possibility values of the resulting fuzzy variable will be frequentist p-values. That is the type of interpretation on which the quantitative correctness of a metric can be successfully argued. Future work on the satellite conjunction analysis problem will include the derivation of that twelve-dimensional solution.

[c]Note that conservatism with respect to an uncertain frequentist proabability is the gold standard for correctness in a probability bounds analysis. This is a different situation. In those problems, probability is unknown but would be accepted as a risk metric if it could be determined precisely. In satellite conjunction analysis, epistemic probability is rejected as a collision risk metric due to probability dilution.

[d]This is to say that, while there is some seed space by which one could call the two-dimensional distribution on the displacement a confidence distribution, the conversion of this two dimensional distribution from random to fuzzy does not result in a consonant confidence structure. This is a complication in the theory of confidence structures which is in need of further exploration.

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